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Exotic relation modules and homotopy types for certain 1–relator groups

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Using stably free non-free relation modules we construct an infinite collection of 2–dimensional homotopy types, each of Euler-characteristic one and with trefoil fundamental group. This provides an affirmative answer to a question asked by Berridge and Dunwoody [1]. We also give new examples of exotic relation modules. We show that the relation module associated with the generating set $\{x, y^4\}$ for the Baumslag–Solitar group $\langle x, y \mid xy^2x^{-1} = y^3\rangle$ is stably free non-free of rank one.

57M20; 57M05

1 Introduction

Given a group G and an integer n the homotopy classification program in dimension two aims to determine all 2–complexes (up to homotopy) with fundamental group G and Euler-characteristic n (see Dunwoody [6], Harlander and Jensen [9], Hog-Angeloni, Metzler and Sieradski [10], Beyl and Waller [2, 3], Dyer and Sieradski [7, 8], Jensen [13] and Johnson [14]). If K is a 2–complex then it is not difficult to see that the Euler-characteristic $\chi(K)$ is bounded from below by $\sum_{i=0}^{2}(-1)^i \dim H_i(G,\mathbb{Q})$, a constant that only depends on the homology of G. Thus we can define $\chi_{\min}(G)$ to be the minimal Euler-characteristic that can occur for a finite 2–complex with fundamental group G. If G is a group of finite geometric dimension 2, that is G is the fundamental group of a finite aspherical 2–complex K, then $\chi_{min}(G) = \chi(K)$ and K is, up to homotopy, the unique 2–complex on the minimal Euler-characteristic level. We show that if G is of geometric dimension 2 and admits a stably free non-free relation module of rank k, then there are at least two homotopically distinct 2–complexes with the same Euler-characteristic, $\chi_{min}(G) + k$ (Corollary 4.3 and Theorem 4.4). We use this to construct an infinite collection of homotopically distinct 2–complexes for the trefoil group, all of Euler characteristic one (Theorem 4.5). This provides an affirmative answer to a question raised by Berridge and Dunwoody [1] (see also Lustig [16] for closely related results).

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These topological applications rely on the existence of non-free relation modules for groups of geometric dimension 2. Let G be a group and $\mathbf{x}$ be a generating set for G. Let F be the free group with basis in one-to-one correspondence to $\mathbf{x}$. The kernel of the canonical map $F \to G$ is denoted by $R(G,\mathbf{x})$ and is called the relation subgroup associated with $\mathbf{x}$. If we abelianize $R = R(G,\mathbf{x})$ we obtain a $\mathbb{Z}G$–module $M(G,\mathbf{x}) = R/[R,R]$,where the G–action is given by conjugation. This module is called the relation module associated with $\mathbf{x}$.

Consider the group G presented by $\langle x, y \mid xy^2x^{-1} = y^3\rangle$. Observe that the elements x and $z = y^4$ also generate G. Indeed, since $xzx^{-1} = y^6$, the element y^2 is in $\langle x, z\rangle$. Since $xy^2x^{-1} = y^3$, we see that y^3 is in $\langle x, z\rangle$ and hence so is y.

Graham Higman observed (see Lyndon and Schupp [18, page 93]) that the generating set consisting of x and z does not support a 1–relator presentation for G. We prove a stronger result.

Theorem 1.1 *Let G be the group defined by $\langle x, y \mid xy^2x^{-1} = y^3\rangle$ and let $z = y^4$. Then the relation module $M(G, \{x, z\})$ cannot be generated by a single element.*

Corollary 1.2 *The relation module $M = M(G, \{x, z\})$ is stably free non-free of rank one: $M \oplus \mathbb{Z}G \approx \mathbb{Z}G^2$.*

Proof Since y is a redundant generator we have that

$$M(G, \{x, y, z\}) \approx M(G, \{x, z\}) \oplus \mathbb{Z}G.$$

Since the generating set $\{x, y, z\}$ supports the aspherical presentation

$$\langle x, y, z | xy^2x^{-1} = y^3, z = y^4\rangle$$

(see Lyndon [17]) it follows that $M(G, \{x, y, z\}) \approx \mathbb{Z}G^2$. □

In [6] Martin Dunwoody shows analogous results for the trefoil group. Later Berridge and Dunwoody [1] show that there are infinitely many non-isomorphic stably free non-free relation modules for the trefoil group, all of rank one. For related results see also Lewin [15].

2 Some combinatorial group theory

Let G be the group presented by $\langle x, y \mid xy^2x^{-1} = y^3\rangle$ and let $z = y^4$.

Lemma 2.1 *The kernel K of the epimorphism*

$$\bar{G} = \langle x, z \mid z = [x, z]^2 \rangle \to G$$

that sends x to x and z to z is non-trivial and free.

Proof Let $\bar{H}$ be the normal closure of z in $\bar{G}$ and let H be the normal closure of z in G. The epimorphism in the statement of the lemma restricts to an epimorphism from $\bar{H}$ to H with kernel K. Indeed, since both G/H and and $\bar{G}/\bar{H}$ are infinite cyclic (generated by x), it follows that the kernel $K\bar{H}/\bar{H}$ of the epimorphism $\bar{G}/\bar{H} \to G/H$ is trivial and so K is contained in $\bar{H}$. Let $z_i = x^i z x^{-i}$, $i \in \mathbb{Z}$. Then $\bar{H}$ has a presentation

$$\bar{H} = \langle z_i \mid z_i = (z_{i+1} z_i^{-1})^2 \rangle, i \in \mathbb{Z}.$$

If we define $u_i = z_{i+1} z_i^{-1}$ then we obtain a presentation

$$\bar{H} = \langle z_i, u_i \mid u_i = z_{i+1} z_i^{-1}, z_i = (z_{i+1} z_i^{-1})^2 \rangle, i \in \mathbb{Z}$$

and hence via Tietze transformations

$$\bar{H} = \langle u_i \mid u_{i+1}^2 = u_i^3 \rangle, i \in \mathbb{Z}.$$

So we see that $\bar{H}$ is an amalgamated product with infinite cyclic vertex groups $\langle u_i \rangle$ and infinite cyclic edge groups that give the relations $u_{i+1}^2 = u_i^3$. Note that the epimorphism $\bar{H} \to H$ sends u_i to $y_i^2 = x^i y^2 x^{-i}$. Since $[y^2, xy^2x^{-1}] = 1$ in G, we see that K contains the element $[u_0, u_1]$, which by the Normal Form Theorem for amalgamated products is non-trivial. It remains to be shown that K is free. Since $\bar{H}$ acts on a tree with infinite cyclic vertex stabilizers conjugate to $\langle u_i \rangle$ and $\langle u_i \rangle K/K$ is the infinite cyclic subgroup $\langle y_i^2 \rangle$ of H, it follows that K intersects the conjugates of $\langle u_i \rangle$ trivially. Hence K acts freely on a tree and hence is free. □

Lemma 2.2 *Let $\mathbf{s}(z, xzx^{-1})$ be a set of relations among z and xzx^{-1} that holds among the generators x and z of G. Let L be the kernel of an epimorphism $\bar{G} = \langle x, z \mid \mathbf{s}(z, xzx^{-1}) \rangle \to G$ that sends x to x and z to z. Then L is not perfect (that is $L/[L, L] \neq 0$).*

Proof Let $\bar{H}$ and H be the normal closures of z in $\bar{G}$ and G, respectively. As in the previous lemma L is also in the kernel of the restriction of the epimorphism to $\bar{H}$. The group $\bar{H}$ has a staggered presentation

$$\bar{H} = \langle z_i \mid \mathbf{s_i}(z_i, z_{i+1}) \rangle, i \in \mathbb{Z}$$

where $\mathbf{s_i} = \mathbf{s_i}(z_i, z_{i+1})$ is a set of relations that hold among the two elements z_i, z_{i+1} of H. Let F be the free group on the z_i, $i \in \mathbb{Z}$, and let S be the normal closure of

the $\bigcup_{i\in\mathbb{Z}} \mathbf{s_i}$, in F. Then $\bar{H} = F/S$ and $L = J/S$ for some normal subgroup J of F. Notice that $H = F/J$ and that J contains the elements $c_i = [z_i, z_{i+1}]$ and $d_i = z_i^{-3}z_{i+1}^2$, $i \in \mathbb{Z}$ (because they present the trivial element in H). We claim that the normal closure of the set $\bigcup_{i\in\mathbb{Z}} \mathbf{s_i} \cup \{c_i, d_i, i \in \mathbb{Z}\}$ in F is the same as the normal closure of $\{e_i = z_i^{-1}(z_{i+1}z_i^{-1})^2,\ i \in \mathbb{Z}\}$ in F. To see this it suffices to show that for every fixed $i \in \mathbb{Z}$ we have

$$^{F(z_i,z_{i+1})}\langle\langle \mathbf{s_i}, c_i, d_i\rangle\rangle = {}^{F(z_i,z_{i+1})}\langle\langle e_i\rangle\rangle.$$

Clearly the right hand side is contained in the left hand side because e_i is a product of conjugates of c_i and d_i (if we are allowed to commute z_i and z_{i+1} then we can turn e_i into d_i). In order to show the other inclusion consider the epimorphism

$$\langle z_i, z_{i+1} \mid e_i\rangle \to \langle z_i, z_{i+1} \mid \mathbf{s_i}, c_i, d_i\rangle.$$

We will show that both groups are infinite cyclic. Hence this epimorphism is an isomorphism, and that settles the claim. Note first that the group on the left is infinite cyclic, generated by $z_{i+1}z_i^{-1}$. Let us consider the group on the right. Notice that the relations $\mathbf{s_i}$, c_i and d_i hold in H (i.e. these elements are contained in J), $\mathbf{s_i}$ by hypothesis and c_i, d_i by direct inspection. Thus we have an epimorphism

$$\langle z_i, z_{i+1} \mid \mathbf{s_i}, c_i, d_i\rangle \to \langle z_i, z_{i+1}\rangle = H_i$$

onto the subgroup H_i of H generated by $z_i = y_i^4 = x^i y^4 x^{-i}$, $z_{i+1} = y_{i+1}^4 = x^{i+1}y^4x^{-(i+1)}$. Since in H we have $y_{i+1}^4 = y_i^6$, we see that $\langle z_i, z_{i+1}\rangle = \langle y_i^4, y_i^6\rangle = \langle y_i^2\rangle$, which is an infinite cyclic subgroup of H. Thus $\langle z_i, z_{i+1} \mid \mathbf{s_i}, c_i, d_i\rangle$ is the image of an infinite cyclic group and has an infinite cyclic image. Hence it is infinite cyclic. This settles the claim.

Let E be the normal closure of $\{e_i,\ i \in \mathbb{Z}\}$ in F and let $K = J/E$. We have just shown that $S \subseteq E$, so K is a homomorphic image of $L = J/S$. So if we assume that L is perfect, we conclude that K is perfect as well. But according to the previous Lemma 2.1 the group K is non-trivial and free. So L can not be perfect. □

3 Some module theory

Proposition 3.1 *Suppose G is a group and $M(G, \mathbf{x})$ is the relation module associated with some generating set $\mathbf{x}$. Suppose furthermore that $\mathbf{s}$ is a subset of $R = R(G, \mathbf{x})$ that gives a set of generators for the $\mathbb{Z}G$–module $M(G, \mathbf{x})$. Let $\tilde{G}$ be the group defined by the presentation $\langle \mathbf{x} \mid \mathbf{s}\rangle$. Then the kernel P of the natural surjection from $\tilde{G}$ onto G is perfect.*

Proof Let F be the free group on $\mathbf{x}$ and let S be the normal closure of $\mathbf{s}$ in F. Since $s[R,R]$, $s \in \mathbf{s}$, generates the relation module we have $R = S[R,R]$. Since $\tilde{G} = F/S$ and $G = F/R$, the kernel of the map $\tilde{G} \to G$ is $P = R/S$. Thus $P/[P,P] = R/S[R,R] = 0$. □

Let G be a group, $F = F(\mathbf{a} \cup \mathbf{b})$ be a free group on the union of sets $\mathbf{a}$ and $\mathbf{b}$ and let $\pi\colon F(\mathbf{a} \cup \mathbf{b}) \to G$ be a group epimorphism. Let $R = R(G, \mathbf{a} \cup \mathbf{b})$ be the kernel of π. Assume that $Q = \pi(F(\mathbf{b}))$ is a free group on basis $\mathbf{b}$ and let H be the normal closure of $\pi(F(\mathbf{a}))$ in G. Note that G is a semi-direct product $H \rtimes Q$. Let $\bar{\mathbf{a}} = \{faf^{-1} \mid f \in F, a \in \mathbf{a}\}$. Then $F(\bar{\mathbf{a}})$, the free group on $\bar{\mathbf{a}}$, is the normal closure of $\mathbf{a}$ in F. Let $\pi'\colon F(\bar{\mathbf{a}}) \to H$ be the restriction of π and let $S = S(H, \bar{\mathbf{a}})$ be the kernel of π'. Note that since $R \subseteq F(\bar{\mathbf{a}})$ we have $S = R$. In particular $M(H, \bar{\mathbf{a}}) = M(G, \mathbf{a} \cup \mathbf{b})$ as $\mathbb{Z}G$–modules, where the G–action on $M(H, \bar{\mathbf{a}})$ is conjugation: $fR * s[S,S] = fsf^{-1}[S,S]$, $f \in F$, $s \in S$.

Let $\mathbf{x}$ be a generating set for the group G. The Cayley-graph $\Gamma(G, \mathbf{x})$ is a graph with vertex set G and edge set $G \times \mathbf{x}$. The initial vertex of the edge (g, x) is g, the terminal vertex is the product gx. The group G acts on this graph via left multiplication and induces $\mathbb{Z}G$–module structures on the homology groups. Every element of $R = R(G, \mathbf{x})$ can be lifted to a closed edge path in the Cayley-graph and this construction is the basis for the Fox-derivative $\mathcal{F}\colon M(G, \mathbf{x}) \to H_1(\Gamma(G, \mathbf{x}))$, $\mathcal{F}(r[R,R]) = \sum_{x \in \mathbf{x}} \frac{\partial r}{\partial x} e_x$, where e_x denotes the edge $(1, x)$. The Fox-derivative can be shown to be a $\mathbb{Z}G$–module isomorphism between the relation module and the first homology of the Cayley-graph (see [18, Chapter II, Section 3]).

Let us now specialize to the situation where $G = \langle x, y \mid xy^2x^{-1} = y^3\rangle$. Let $z = y^4$, $\mathbf{a} = \{z\}$, $\mathbf{b} = \{x\}$. Then $\bar{\mathbf{a}} = \{z_i \mid i \in \mathbb{Z}\}$, where $z_i = x^i z x^{-i}$. We have $M(G, \{x, z\}) = M(H, \{z_i\}_{i \in \mathbb{Z}})$ as $\mathbb{Z}G$–modules. Now

$$H_1(\Gamma(H, \{z_i\}_{i\in\mathbb{Z}})) = \ker(\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}He_i \xrightarrow{\partial} \mathbb{Z}H),$$

where e_i is the edge $(1, z_i)$ and hence $\partial(e_i) = z_i - 1$. Since $G = H \rtimes \langle x\rangle$, the group ring $\mathbb{Z}G$ is a skewed Laurent-polynomial ring $\mathbb{Z}H[x^{\pm 1}]$. Furthermore $\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}He_i$ is isomorphic to $\mathbb{Z}G = \mathbb{Z}H[x^{\pm 1}]$, the isomorphism sending e_i to x^i and the map $\partial\colon \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}He_i \to \mathbb{Z}H$, $\partial(x^i) = z_i - 1$, is a $\mathbb{Z}G$–module homomorphism (the G–action on $\mathbb{Z}H$ is $hx^i * h' = hx^i h' x^{-i}$). In particular $H_1(\Gamma(H, \{z_i\}_{i\in\mathbb{Z}}))$ is a $\mathbb{Z}G$–module and the Fox-derivative $\mathcal{F}\colon M(H, \{z_i\}_{i\in\mathbb{Z}}) \to H_1(\Gamma(H, \{z_i\}_{i\in\mathbb{Z}}))$ is a $\mathbb{Z}G$–module isomorphism.

If $\alpha = \alpha_j x^j + \ldots + \alpha_{j+n} x^{j+n} \in ZH[x^{\pm 1}]$, α_j, α_{j+n} both not zero, then we call n the length of α, $n = l(\alpha)$. Note that if $\alpha = \beta\gamma$ then $l(\alpha) = l(\beta) + l(\gamma)$. This uses the

fact that, because G is a torsion-free 1–relator group, the group ring $\mathbb{Z}G$ has no zero divisors (see Brodskii [4], Howie [11, 12]).

Lemma 3.2 *Suppose $\alpha = \alpha_0 e_0 + \ldots + \alpha_n e_n$ is an element of $H_1(\Gamma(H, \{z_i\}_{i\in\mathbb{Z}})) \subseteq \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}He_i$. Then there exist elements $s_1, \ldots, s_m \in R = R(H, \{z_i\}_{i\in\mathbb{Z}})$ and $f_1, \ldots, f_m \in F(\{z_i\}_{i\in\mathbb{Z}})$ such that each s_j, $j = 1, \ldots, m$, is a word in the letters $z_0, \ldots, z_n$ and $\mathcal{F}(\prod_{k=0}^{m} f_k s_k f_k^{-1}[R,R]) = \alpha$.*

Proof The cycle α is a sum of edges in $\Gamma(H, \{z_i\}_{i\in\mathbb{Z}})$ with edge labels involving only letters from $\{z_0, \ldots, z_n\}$. These edges can be arranged to form closed edge paths $P_1, \ldots, P_m$. Choose paths $Q_1, \ldots, Q_m$ such that Q_i connects the vertex 1 to a vertex occurring in P_i. Reading off the edge labels on the path $Q_i P_i Q_i^{-1}$, $i = 1, \ldots, m$, gives a word of the form $f_i s_i f_i^{-1}$, where $s_i \in R$ involves only letters from $\{z_0, \ldots, z_n\}$, and $\mathcal{F}(\prod_{k=0}^{m} f_k s_k f_k^{-1}[R,R]) = \alpha$. □

Proof of Theorem 1.1 Suppose $M(G, \{x, z\})$ is generated (as $\mathbb{Z}G$–module) by a single element. Then so is $M(H, \{z_i\}_{i\in\mathbb{Z}})$. Hence $H_1(\Gamma(H, \{z_i\}_{i\in\mathbb{Z}}))$ is also singly generated, say by α. Note that $z_1^2 z_0^{-3} \in R(H, \{z_i\}_{i\in\mathbb{Z}})$, so $\beta = \mathcal{F}(z_1^2 z_0^{-3}[R,R])$ is an element of length one in $H_1(\Gamma(H, \{z_i\}_{i\in\mathbb{Z}}))$ and $\beta = \gamma\alpha$ for some $\gamma \in \mathbb{Z}G = \mathbb{Z}H[x^{\pm 1}]$. Since $l(\beta) = l(\gamma) + l(\alpha)$, we conclude that the length of α is less or equal to one, so we may assume $\alpha = \alpha_0 e_0 + \alpha_1 e_1$. It follows from Lemma 3.2 that there are elements $s_1(z_0, z_1), \ldots, s_m(z_0, z_1)$ in $F(z_0, z_1)$ that give rise to $\mathbb{Z}G$–module generators for $M(H, \{z_i\}_{i\in\mathbb{Z}})$. Thus the set $\mathbf{s}(z, xzx^{-1}) = \{s_0(z, xzx^{-1}), \ldots, s_m(z, xzx^{-1})\}$ generates the $\mathbb{Z}G$–module $M(G, \{x, z\})$. Proposition 3.1 implies that the kernel of the map

$$\bar{G} = \langle x, z \mid \mathbf{s}(z, xzx^{-1})\rangle \to G$$

that sends x to x and z to z is perfect. This contradicts Lemma 2.2. □

4 Topological applications

In the last section we have seen that the relation module $M = M(G, \{x, y^4\})$ for the group $G = \langle x, y \mid xy^2x^{-1} = y^3\rangle$ is not generated by a single element, hence is certainly not isomorphic to $\mathbb{Z}G$. In this section we will use this fact to exhibit 2–complexes with fundamental group G and the same Euler-characteristic that are not homotopically equivalent.

Let X be a CW–complex which is the union of a family of non-empty subcomplexes X_α, where α ranges over some index set J. Let $\mathcal{N}$ be the nerve associated with this

covering of X. The nerve $\mathcal{N}$ is a simplicial complex with vertex set J. The n–simplices are subsets $\{\alpha_1, ..., \alpha_n\}$, $\alpha_i \in J$, $i = 1, ..., n$, such that the intersection $X_{\alpha_1} \cap ... \cap X_{\alpha_n}$ is not empty.

The following result is an immediate consequence of the Mayer–Vietoris spectral sequence (see Brown [5, page 166]).

Lemma 4.1 *If $\mathcal{N}$ is a tree we have a long exact sequence*

$$... \to H_{p+1}(X) \to \bigoplus_{\{\alpha,\alpha'\}\in\mathcal{N}^{(1)}} H_p(X_\alpha \cap X_{\alpha'}) \to \bigoplus_{\alpha\in\mathcal{N}^{(0)}} H_p(X_\alpha) \to H_p(X) \to ...$$

Consider the amalgamated product of groups $G = G_1 *_H G_2$. Let K_i be an Eilenberg–MacLane complex for G_i, $i = 1, 2$, and L be an Eilenberg–MacLane complex for H. For convenience we assume that each of these complexes have a single vertex. Let u, v, w be the vertices of K_1, K_2, L, respectively. An Eilenberg–MacLane complex $K = K_1 \cup (L \times [0, 1]) \cup K_2$ for G is obtained from K_1, $L \times [0, 1]$, and K_2 by gluing $L \times \{0\}$ to K_1 via a map induced by the inclusion $H \hookrightarrow G_1$ and gluing $L \times \{1\}$ to K_2 via a map induced by the inclusion $H \hookrightarrow G_2$.

Theorem 4.2 (a) *For $n > 2$ there is a short exact sequence*

$$0 \to \mathbb{Z}G \otimes_{G_1} \pi_n(K_1^{(n)}) \oplus \mathbb{Z}G \otimes_{G_2} \pi_n(K_2^{(n)}) \to \pi_n(K^{(n)}) \to \mathbb{Z}G \otimes_H \pi_{n-1}(L^{(n-1)}) \to 0.$$

(b) *For $n = 2$ there is a short exact sequence*

$$0 \to \mathbb{Z}G \otimes_{G_1} \pi_2(K_1^{(2)}) \oplus \mathbb{Z}G \otimes_{G_2} \pi_2(K_2^{(2)}) \to \pi_2(K^{(2)}) \to \mathbb{Z}G \otimes_H M(H, \mathbf{x}) \to 0,$$

where $\mathbf{x}$ is the generating set for H coming from the 1–skeleton of L.

Proof Let X be the n–skeleton of the universal covering $\tilde{K}$ and let $p\colon \tilde{K} \to K$ be the covering projection. Let K_u be the n–skeleton of $K_1 \cup (L \times [0, \frac{1}{2}])$ and K_v be the n–skeleton of the other half, $K_2 \cup (L \times [\frac{1}{2}, 1])$. Note that $K_u \cap K_v = L^{(n-1)} \times \{\frac{1}{2}\}$. Let $\tilde{w}$ be a fixed lift of the point $\{w\} \times \{\frac{1}{2}\}$. Let X_u be the component of $p^{-1}(K_u)$ that contains $\tilde{w}$ and X_v be the component of $p^{-1}(K_v)$ that contains $\tilde{w}$. The intersection $Y = X_u \cap X_v$ is a component of $p^{-1}(K_u \cap K_v) = p^{-1}(L^{(n-1)} \times \{\frac{1}{2}\})$ and hence is homeomorphic to the $(n-1)$–skeleton of the universal covering of L. Note that $p^{-1}(K_u)$ is the disjoint union of the translates gX_u, where g is taken from $T(G/G_1)$, a transversal for G/G_1. Analogously, $p^{-1}(K_v)$ is the disjoint union of the translates $g'X_u$, where g' is taken from $T(G/G_2)$. Let $\mathcal{N}$ be the nerve associated with the covering of X by the components of $p^{-1}(K_u)$ and $p^{-1}(K_v)$. This nerve is a simplicial tree, isomorphic to the Bass–Serre

tree associated with the amalgamated product $G_1 *_H G_2$. The vertices of this tree are the translates gX_u for $g \in T(G/G_1)$ and $g'X_v$ for $g' \in T(G/G_2)$. The intersection $gX_u \cap g'X_v \neq \emptyset$ if and only if there is a $g'' \in G$ so that $gX_u = g''X_u$ and $g'X_v = g''X_v$. In this case $gX_u \cap g'X_v = g''(X_v \cap X_v) = g''Y$. Hence the edges of $\mathcal{N}$ are the translates $g''Y$, $g'' \in T(G/H)$. We apply the previous lemma and obtain a long exact sequence

$$\cdots \to H_{n+1}(X) \to \bigoplus_{g'' \in T(G/H)} H_n(g''Y) \to$$
$$\to \bigoplus_{g \in T(G/G_1)} H_n(gX_u) \bigoplus_{g' \in T(G/G_2)} H_n(g'X_v) \to H_n(X) \to \cdots$$

Since Y is $(n-1)$–dimensional and both X_u, X_v are n–dimensional and $(n-1)$–connected, this yields the short exact sequence

$$0 \to \bigoplus_{g \in T(G/G_1)} H_n(gX_u) \oplus \bigoplus_{g' \in T(G/G_2)} H_n(g'X_v) \to H_n(X) \to \bigoplus_{g'' \in T(G/H)} H_{n-1}(g''Y) \to 0.$$

Let us first assume that $n > 2$. Since X_u is the n–skeleton of the universal covering of the Eilenberg–MacLane complex K_u (which is homotopically equivalent to K_1) we have $H_n(X_u) \approx \pi_n(X_u) \approx \pi_n(K_1^{(n)})$ by the Hurewicz Theorem. So

$$\bigoplus_{g \in T(G/G_1)} H_n(gX_u) \approx \mathbb{Z}G \otimes_{G_1} \pi_n(K_1^{(n)})$$

as $\mathbb{Z}G$–modules. By analogous arguments we have

$$\bigoplus_{g' \in T(G/G_2)} H_n(g'X_v) \approx \mathbb{Z}G \otimes_{G_2} \pi_n(K_2^{(n)})$$

and

$$\bigoplus_{g'' \in T(G/H)} H_{n-1}(g''Y) \approx \mathbb{Z}G \otimes_H \pi_{n-1}(L^{(n-1)}).$$

So the above short exact sequence, after making the isomorphic replacements, yields the short exact sequence exhibited in statement (a).

Let us assume now that $n = 2$. In that case Y is the 1–skeleton of the universal covering of L and hence is the Cayley-graph of the group H, associated with the generating set $\mathbf{x}$ that arises from the 1–skeleton of L (recall that we assumed L to have a single vertex, so the 1–skeleton is a wedge of circles). Thus $H_1(Y) \approx H_1(\Gamma(H, \mathbf{x})) \approx M(H, \mathbf{x})$ and

$$\bigoplus_{g'' \in T(G/H)} H_1(g''Y) \approx \mathbb{Z}G \otimes_H M(H, \mathbf{x}).$$

The above short exact sequence, after making the isomorphic replacements, yields the short exact sequence exhibited in statement (b). □

A 2–complex K is *aspherical* if $\pi_2(K) = 0$. A group is *aspherical* if it is the fundamental group of an aspherical 2–complex. A consequence of the above theorem is that for aspherical groups every relation module is also a second homotopy module. Indeed, assume J is an aspherical 2–complex with fundamental group G (in particular J is an Eilenberg–MacLane complex). Suppose $M = M(G, \mathbf{x})$ is a relation module for G associated with some generating set. Let L be an Eilenberg–MacLane complex for G with 1–skeleton a bouquet of circles in one-to-one correspondence with the elements of $\mathbf{x}$. Writing G as an amalgamated product $G = G *_G G$ we apply the above construction and build an Eilenberg–MacLane complex $K = K_1 \cup (L \times [0,1]) \cup K_2$ with $K_1 = J$, $K_2 = J$. The ends $L \times \{0\}$ and $L \times \{1\}$ are attached to the two copies of J via a map induced by the identity map from G to G. Theorem 4.2(b) implies the following result.

Corollary 4.3 *Let J be an aspherical 2–complex with fundamental group G. Let $M(G, \mathbf{x})$ be a relation module associated with some generating set $\mathbf{x}$ and let L be an Eilenberg–MacLane complex with 1–skeleton a bouquet of circles in one-to-one correspondence with $\mathbf{x}$. Then the 2–complex $K^{(2)} = (J \cup (L \times [0,1]) \cup J)^{(2)}$ has Euler-characteristic $\chi(K^{(2)}) = 2\chi_{min}(G) - 1 + |\mathbf{x}|$ and the second homotopy module $\pi_2(K^{(2)})$ is isomorphic to the relation module $M(G, \mathbf{x})$.*

Consider the group G presented by $\langle x, y \mid xy^2x^{-1} = y^3 \rangle$. Let J be the 2–complex built from this presentation. Since J is aspherical (see Lyndon [17]) we have $\chi_{min}(G) = \chi(J) = 0$ and J is the only 2–complex (up to homotopy equivalence) on the minimal Euler-characteristic level. The next result shows that the situation is different on the next level, $\chi_{min}(G) + 1$.

Theorem 4.4 *Let J be the 2–complex built on the presentation*

$$\langle x, y \mid xy^2x^{-1} = y^3 \rangle$$

for G. Let K be the 2–complex built on the presentation

$$\langle x, y, x', y' \mid xy^2x^{-1} = y^3, x'y'^2x'^{-1} = y'^3, x = x', y^4 = y'^4 \rangle$$

for G. Then $J \vee S^2$ and K are homotopically distinct 2–complexes with fundamental group G and Euler-characteristic $\chi_{min}(G) + 1$.

Proof We write G as an amalgamated product $G = G *_G G$ and build an Eilenberg–MacLane complex $K' = J \cup (L \times [0,1]) \cup J$, where we take for L an Eilenberg–MacLane complex with 1–skeleton a bouquet of two circles corresponding to the generating set $\{x, y^4\}$. By Corollary 4.3, $\pi_2(K'^{(2)})$ is isomorphic to $M(G, \{x, y^4\})$ and hence is

different from $\mathbb{Z}G$. Note that K' can be built so that, after collapsing a maximal tree in the 1–skeleton of $K'^{(2)}$ (which consists of a single edge) we obtain the complex K. Since $\pi_2(J \vee S^2) = \mathbb{Z}G$ the desired result follows. □

In [1] P H Berridge and M Dunwoody give an infinite sequence of pairwise distinct stably free non-free relation modules for the trefoil group G and ask the question whether this can be used to construct infinitely many pairwise non-homotopic 2–complexes with identical Euler-characteristic and fundamental group G. The above theorem answers this question affirmatively. Consider the trefoil group presented by $\langle x, y \mid x^2 = y^3 \rangle$. Let $\mathbf{x_i} = \{x^{2i+1}, y^{3i+1}\}$, $i \in \mathbb{N}$. The set of relation modules $\{M(G, \mathbf{x_i}) \mid i \in \mathbb{N}\}$ contains infinitely many non-isomorphic stably free non-free modules of rank one [1].

Theorem 4.5 *Let G be the trefoil group presented by $\langle x, y \mid x^2 = y^3 \rangle$. Let K_i be the 2–complex built on the presentation*

$$\langle x, y, x', y' \mid x^2 = y^3, x'^2 = y'^3, x^{2i+1} = x'^{2i+1}, y^{3i+1} = y'^{3i+1} \rangle$$

for G. The set $\{K_i \mid i \in \mathbb{N}\}$ contains infinitely many homotopically distinct 2–complexes with fundamental group G and Euler-characteristic equal to one.

In [16] M Lustig constructs an infinite collection of 2–dimensional homotopy types with the same fundamental group and Euler characteristic. However the fundamental group in these examples is distinct from the trefoil group. Theorem 4.5 answers the precise question raised by Berridge and Dunwoody [1] in the affirmative.

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